Machine Learning for Inverse Problems in Computational Engineering

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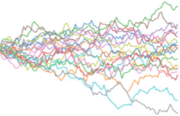
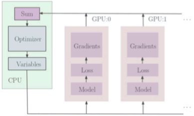
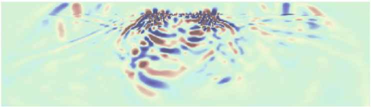
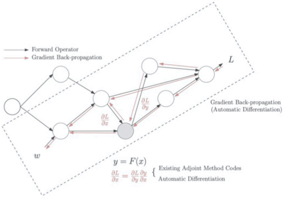
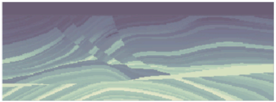
https://github.com/kailaix/ADCME.jl

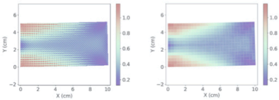








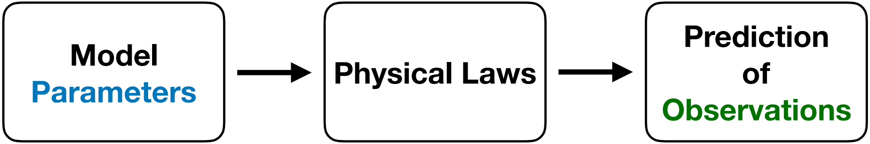




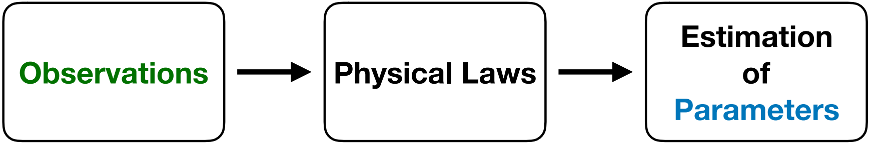
Outline

Inverse Modeling

Forward Problem



Inverse Problem



Inverse Modeling

We can formulate inverse modeling as a PDE-constrained optimization problem

$$\min_{\theta} L_h(u_h) \quad \text{s.t. } F_h(\theta, u_h) = 0$$

- The loss function L_h measures the discrepancy between the prediction u_h and the observation u_{obs}, e.g., $L_h(u_h) = \|u_h - u_{\text{obs}}\|_2^2$.
- θ is the model parameter to be calibrated.
- The physics constraints $F_h(\theta, u_h) = 0$ are described by a system of partial differential equations. Solving for u_h may require solving linear systems or applying an iterative algorithm such as the Newton-Raphson method.

Function Inverse Problem

$$\min_{f} L_h(u_h) \quad \text{s.t.}\ F_h(f, u_h) = 0$$

What if the unknown is a function instead of a set of parameters?

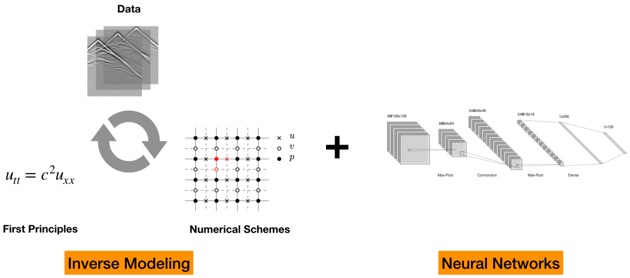
- Koopman operator in dynamical systems.
- Constitutive relations in solid mechanics.
- Turbulent closure relations in fluid mechanics.
- ...

The candidate solution space is infinite dimensional.

Machine Learning for Computational Engineering

$$\min_{\theta} L_h(u_h) \quad \text{s.t. } F_h(NN_\theta, u_h) = 0$$

- Deep neural networks exhibit capability of approximating high dimensional and complicated functions.
- **Machine Learning for Computational Engineering**: the unknown function is approximated by a deep neural network, and the physical constraints are enforced by numerical schemes.
- Satisfy the physics to the largest extent.

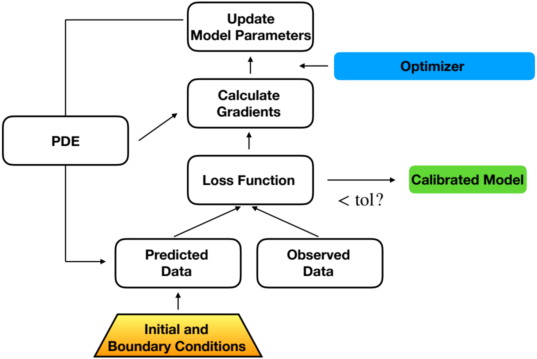


Gradient Based Optimization

$$\min_{\theta} L_h(u_h) \quad \text{s.t. } F_h(\theta, u_h) = 0 \tag{1}$$

- We can now apply a gradient-based optimization method to (1).
- The key is to calculate the gradient descent direction g^k

$$\theta^{k+1} \leftarrow \theta^k - \alpha g^k$$



Outline

Automatic Differentiation

The fact that bridges the technical gap between machine learning and inverse modeling:

- Deep learning (and many other machine learning techniques) and numerical schemes share the same computational model: composition of individual operators.

Mathematical Fact

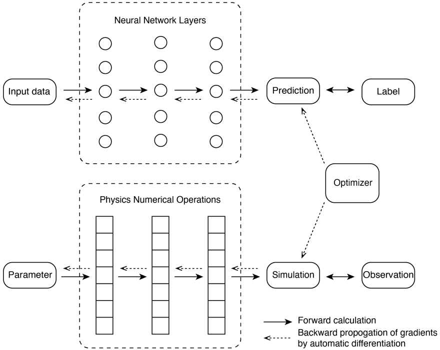
Back-propagation

||

Reverse-mode Automatic Differentiation

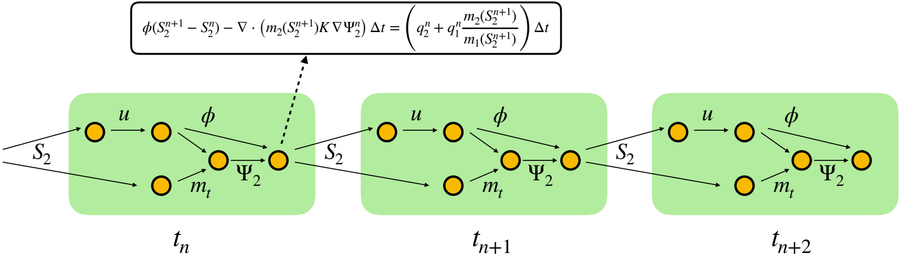
||

Discrete Adjoint-State Method

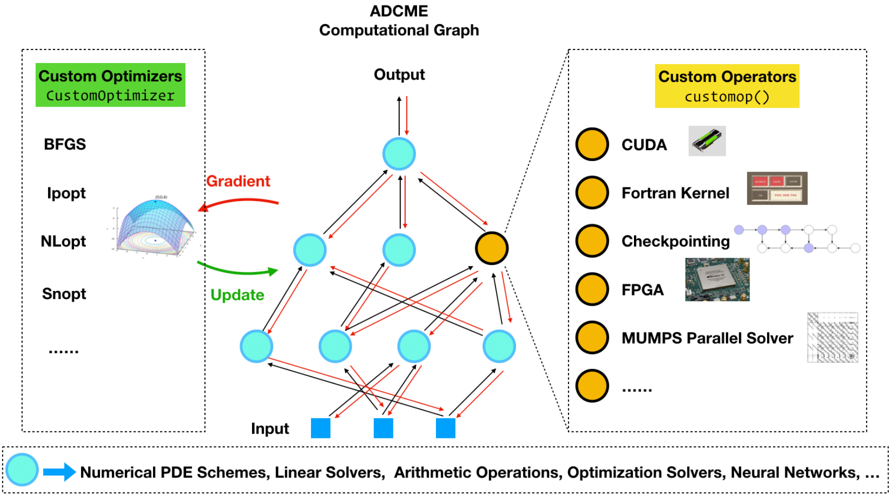


Computational Graph for Numerical Schemes

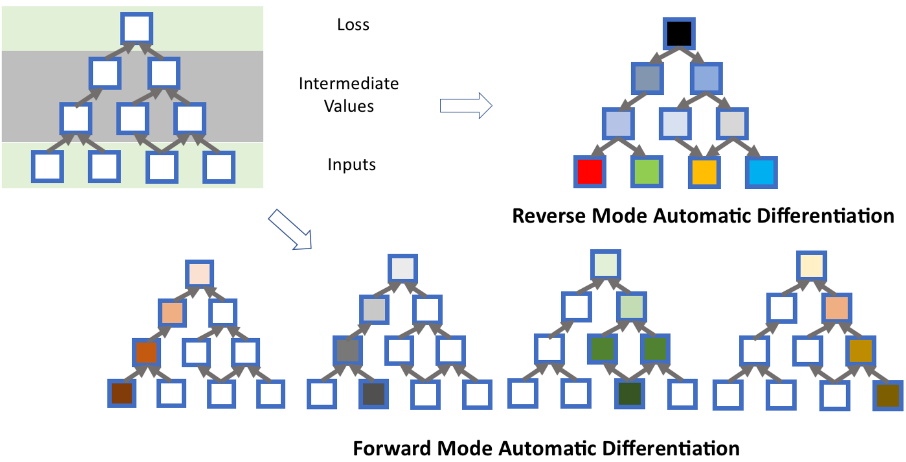
- To leverage automatic differentiation for inverse modeling, we need to express the numerical schemes in the "AD language": computational graph.
- No matter how complicated a numerical scheme is, it can be decomposed into a collection of operators that are interlinked via state variable dependencies.



ADCME: Computational-Graph-based Numerical Simulation



Automatic Differentiation: Forward-mode and Reverse-mode



What is the Appropriate Model for Inverse Problems?

- In general, for a function $f : \mathbb{R}^n \to \mathbb{R}^m$

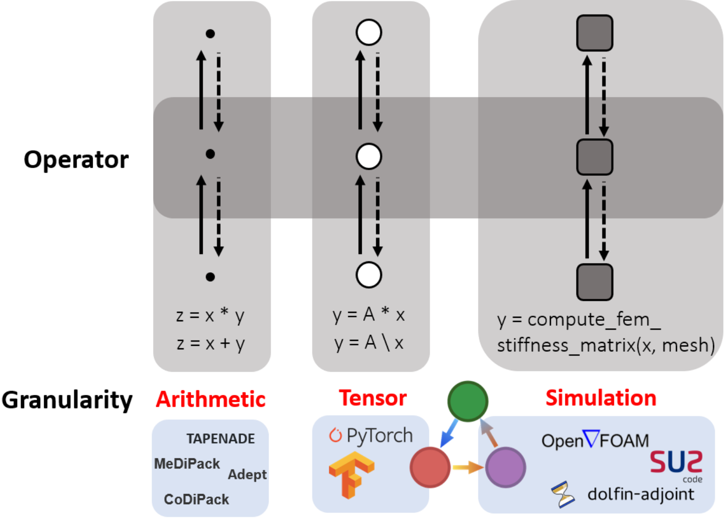
Mode	Suitable for ...	Complexity[1]	Application
Forward	$m \gg n$	$\leq 2.5\ \mathrm{OPS}(f(x))$	UQ
Reverse	$m \ll n$	$\leq 4\ \mathrm{OPS}(f(x))$	Inverse Modeling

- There are also many other interesting topics
  - Mixed mode AD: many-to-many mappings.
  - Computing sparse Jacobian matrices using AD by exploiting sparse structures.

Margossian CC. A review of automatic differentiation and its efficient implementation. Wiley Interdisciplinary Reviews: Data Mining and Knowledge Discovery. 2019 Jul;9(4):e1305.

[1] OPS is a metric for complexity in terms of fused-multiply adds.

Granularity of Automatic Differentiation

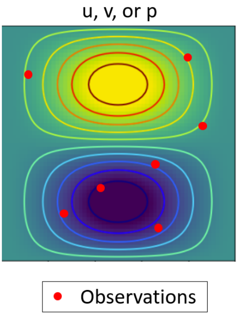


Outline

Inverse Modeling of the Stokes Equation

- The governing equation for the Stokes problem

$$\begin{aligned} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \quad \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 & \quad \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \quad \text{on } \partial\Omega \end{aligned}$$



- The weak form is given by

$$\begin{aligned} (\nu \nabla u, \nabla v) - (p, \nabla \cdot v) &= (f, v) \\ (\nabla \cdot u, q) &= 0 \end{aligned}$$

Inverse Modeling of the Stokes Equation

```
nu = Variable(0.5)
K = nu*constant(compute_fem_laplace_matrix(m, n, h))
B = constant(compute_interaction_matrix(m, n, h))
Z = [K -B'
    -B spdiag(zeros(size(B,1)))]

# Impose boundary conditions
bd = bcnode("all", m, n, h)
bd = [bd; bd .+ (m+1)*(n+1); ((1:m) .+ 2(m+1)*(n+1))]
Z, _ = fem_impose_Dirichlet_boundary_condition1(Z, bd, m, n, h)

# Calculate the source term
F1 = eval_f_on_gauss_pts(f1func, m, n, h)
F2 = eval_f_on_gauss_pts(f2func, m, n, h)
F = compute_fem_source_term(F1, F2, m, n, h)
rhs = [F;zeros(m*n)]
rhs[bd] .= 0.0

sol = Z\rhs
```

Inverse Modeling of the Stokes Equation

- The distinguished feature compared to traditional forward simulation programs: the model output is differentiable with respect to model parameters!

```
loss = sum((sol[idx] - observation[idx])^2)
g = gradients(loss, nu)
```

- Optimization with a one-liner:

```
BFGS!(sess, loss)
```

PoreFlow/ADCME

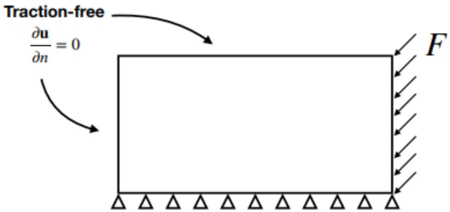
Simulation Program

Outline

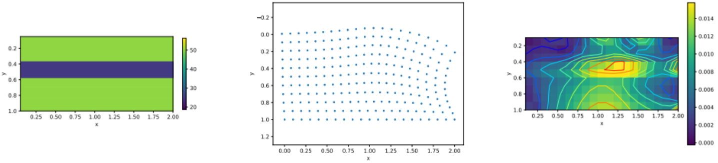
Linear Poroelasticity

- The governing equation for linear poroelasticity with a spatially-varying viscosity coefficient

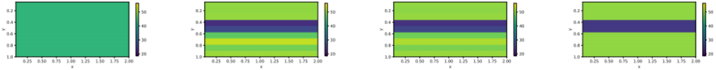
$$
\begin{aligned}
\mathrm{div}\boldsymbol{\sigma} + \rho\mathbf{g} &= \ddot{\mathbf{u}} \\
\dot{\sigma}_{ij} + \frac{\mu}{\eta(y)}\left(\sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}\right) &= 2\mu\dot{\epsilon}_{ij} + \lambda\dot{\epsilon}_{kk}\delta_{ij}
\end{aligned}
$$



- The true model of $\eta(y)^{-1}$, the displacement at the terminal time, and the von Mises stress distribution at the terminal time.



- Evolution of learned $\eta(y)^{-1}$ at iteration 0, 80, 160, and 240.



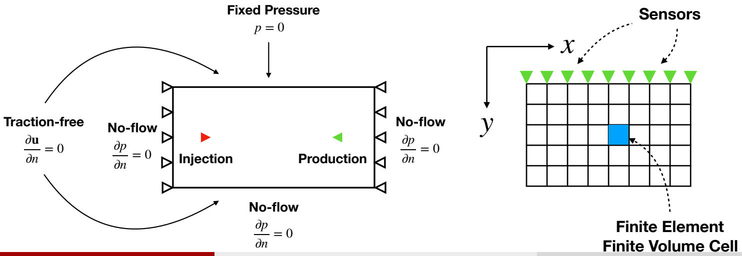
Viscoelasticity

- Multi-physics Interaction of Coupled Geomechanics and Multi-Phase Flow Equations

$$
\begin{aligned}
\mathrm{div}\boldsymbol{\sigma}(\mathbf{u}) - b\nabla p &= 0\\
\frac{1}{M}\frac{\partial p}{\partial t} + b\frac{\partial \epsilon_v(\mathbf{u})}{\partial t} - \nabla\cdot\left(\frac{k}{B_f\mu}\nabla p\right) &= f(x,t)\\
\boldsymbol{\sigma} &= \boldsymbol{\sigma}(\boldsymbol{\epsilon}, \dot{\boldsymbol{\epsilon}})
\end{aligned}
$$

- Approximate the constitutive relation by a neural network

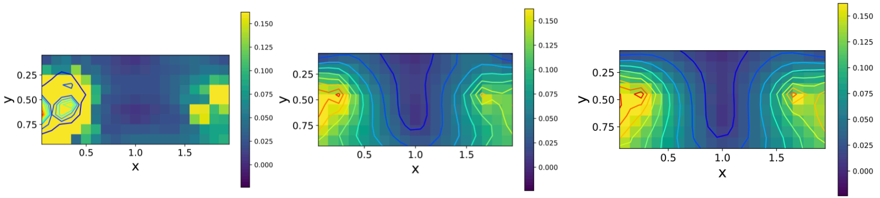
$$
\boldsymbol{\sigma}^{n+1} = \mathcal{N}\mathcal{N}_{\boldsymbol{\theta}}(\boldsymbol{\sigma}^n, \boldsymbol{\epsilon}^n) + H\boldsymbol{\epsilon}^{n+1}
$$



Viscoelasticity

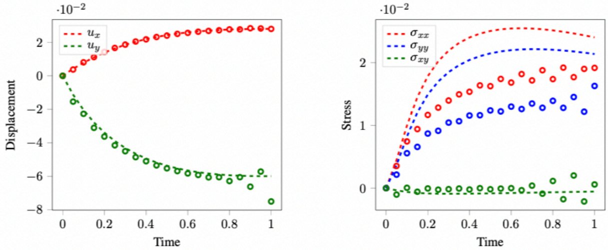
- Comparison with space varying linear elasticity approximation

$$\boldsymbol{\sigma} = H(x, y)\epsilon$$

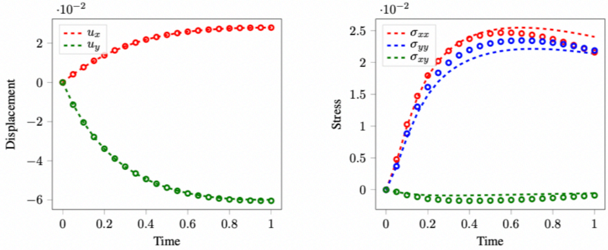


Space Varying Linear Elasticity | **NN** | **True**

Viscoelasticity



(a) Space Varying Linear Elasticity



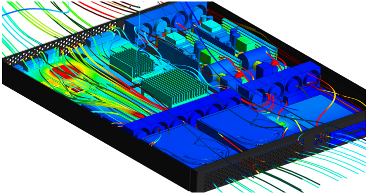
(b) NN-based Viscoelasticity

Navier-Stokes Equation

- Steady-state Navier-Stokes equation

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla)\mathbf{u} &= -\frac{1}{\rho}\nabla p + \nabla\cdot(\nu\nabla\mathbf{u}) + \mathbf{g}\\
\nabla\cdot\mathbf{u} &= 0
\end{aligned}
$$

- Inverse problem are ubiquitous in fluid dynamics:



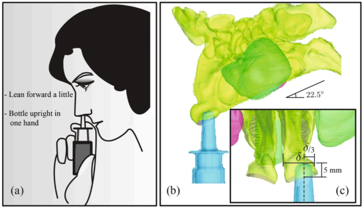
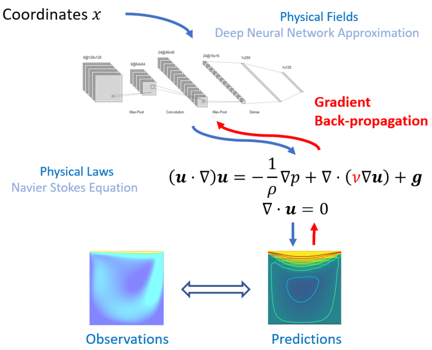
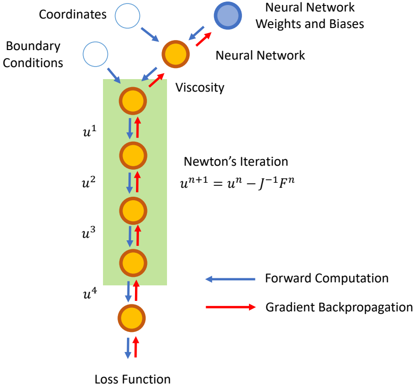


Figure: Left: electronic cooling; right: nasal drug delivery.

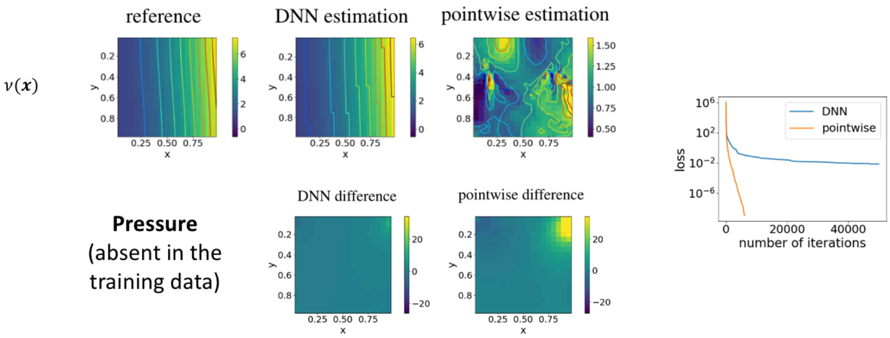
Navier-Stokes Equation



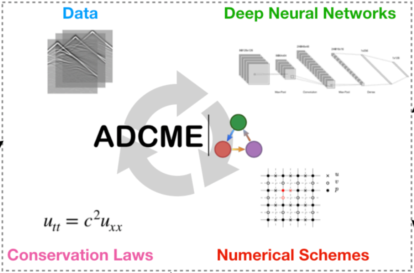


Navier-Stokes Equation

- Data: (u, v)
- Unknown: $\nu(\mathbf{x})$ (represented by a deep neural network)
- Prediction: p (absent in the training data)
- The DNN provides regularization, which generalizes the estimation better!



A General Approach to Inverse Modeling



PoreFlow.jl
Geomechanics
Viscoelasticity
Multiphase Flow
Multiphysics
* coming soon

FwiFlow.jl
Multiphase Flow
Nonlocal Operators
https://github.com/lidongzh/FwiFlow.jl

ADSeismic.jl
General Seismic Inversion
https://github.com/kailaix/ADSeismic.jl

NNFEM.jl
Constitutive Law Modeling
Hyperelasticity
Elasto-Plasticity
https://github.com/kailaix/NNFEM.jl