p-norm cones and second-order cone representations

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In this note, we review how a few power inequalities, and related p-norm constraints for (rational) $p \in (1, \infty)$, may be represented as second order cone programs. These results follow Alizadeh and Goldfarb [1], who enumerate the reductions below and several more such inequalities.

Our starting point is to note that given a vector $w \in \mathbb{R}^n$ and points $x, y \geq 0$, the constraint

$$\|w\|_2 \leq xy$$

is equivalent to the constraint

$$\left\| \begin{bmatrix} 2w \\ x - y \end{bmatrix} \right\|_2 \leq x + y, \quad x \geq 0, y \geq 0, \tag{1}$$

which is evident by squaring both sides and rearranging. Thus, we also obtain that the set of t, x, y such that $x, y \geq 0$ and $t^2 \leq xy$ is representable as a second-order cone. This insight allows us to represent more complicated powers as second-order cones.

Products as second order cones

Consider the set of $t \in \mathbb{R}, s \in \mathbb{R}_+^{2^k}$ such that

$$t^{2^k} \leq s_1 s_2 \cdots s_{2^k}, \quad s_i \geq 0, \text{ all } s_i. \tag{2}$$

We claim this set is SOCP representable. Indeed, introduce variables

$$\left\{u_{i,j} : j \in \{1, \ldots, k-1\}, i \in \{1, \ldots, 2^j\}, u_{i,j} \geq 0\right\}.$$

Then it is clear that the constraint (2) is equivalent to

$$t^{2^k} \leq u_{1,k-1}^2 u_{2,k-1}^2 \cdots u_{2^{k-1},k-1}^2, \quad u_{i,k-1}^2 \leq s_{2i-1} s_{2i}, \quad \text{all } i,$$

or

$$t^{2^{k-1}} \leq u_{1,k-1} u_{2,k-1} \cdots u_{2^{k-1},k-1}, \quad u_{i,k-1}^2 \leq s_{2i-1} s_{2i}, \quad \text{all } i \in \{1, \ldots, 2^{k-1}\}.$$

Recursively applying this construction through levels $j = k-2, \ldots, 1$, we obtain the set of inequalities

$$\begin{aligned} t^2 \leq u_{1,1} u_{2,1}, \ \ u_{i,j-1}^2 \leq u_{2i-1,j} u_{2i,j} \text{ for } \ j \in \{2, \ldots, k-2\}, i \in \{1, \ldots, 2^{j-1}\} \\ u_{i,k-1}^2 \leq s_{2i-1} s_{2i} \ \text{ for } i \in \{1, \ldots, 2^{k-1}\}, \end{aligned} \tag{3}$$

where all $u_{i,j}$ are non-negative. Each of these inequalities, by the representation (1), corresponds to a second-order cone in $\mathbb{R}_+^3$, and we have introduced $2^k - 1$ such inequalities.

Product inequalities as second order cones

We now provide reductions of inequalities of the more restrictive form

$$x^n \leq t^{p_1} s^{p_2}, \quad x, t, s \geq 0 \tag{4}$$

where $p_1 + p_2 = n$ and $p_1, p_2, n \in \mathbb{N}$, into a sequence of second-order cone-represented sets. This is the building block for representations of general p-norm inequalities to come. In particular, we develop a procedure that takes inequalities of one of the three forms

$$x^n \le t^{p_1} s^{p_2} \tag{5a}$$
$$x^n \le t^{p_1} s^{p_2} u \tag{5b}$$
$$x^n \le t^{p_1} s^{p_2} u^2 \tag{5c}$$

and recurses with a power $\le (n+1)/2$ on x and one of the forms (5). Note that if $n = 2$, then we already have an immediate second order cone representation (1).

(1) The case with u^0, inequality (5a). We have two possibilities: either n is even or n is odd.

a. n is even: we assume that $n \ge 4$, as otherwise we have the inequality $x^2 \le ts$, which is trivial. In this case, either both p_1 and p_2 are even or both are is odd. If they are even, inequality (5a) is equivalent to $x^{n/2} \le t^{p_1/2} s^{p_2/2}$, which gives us a recursive step. If p_1 is odd, then p_2 is odd, and inequality (5a) is equivalent to the pair

$$x^n \le t^{p_1-1} s^{p_2-1} u^2, \quad u^2 \le ts, \quad \text{or} \quad x^{n/2} \le t^{\frac{p_1-1}{2}} s^{\frac{p_2-1}{2}} u, \quad u^2 \le ts,$$

which allows us to recurse to case (5b) with lower powers on s, t, and x.

b. n is odd: In this case, we have exactly one of p_1 and p_2 is odd; assume w.l.o.g. that p_1 is odd. Then inequality (5a) is equivalent to $x^{n+1} \le t^{p_1-1} s^{p_2} xt$, and introducing the variable $u \ge 0$, we have the equivalent representation

$$x^{n+1} \le t^{p_1-1} s^{p_2} u^2, \quad u^2 \le xt, \quad \text{or} \quad x^{\frac{n+1}{2}} \le t^{\frac{p_1-1}{2}} s^{\frac{p_2}{2}} u, \quad u^2 \le xt.$$

This allows us to recurse to case (5b).

(2) The case u, inequality (5b). We again have two possibilities: either n is even or n is odd.

a. n is even: In this case, exactly one of p_1 and p_2 is odd; assume w.l.o.g. that p_1 is odd. Then we have the equivalent representation

$$x^n \le t^{p_1-1} s^{p_2} w^2, \quad w^2 \le tu, \quad \text{or} \quad x^{\frac{n}{2}} \le t^{\frac{p_1-1}{2}} s^{\frac{p_2}{2}} w, \quad w^2 \le tu.$$

This is again an inequality of the form (5b).

b. n is odd: In this case, either both p_1 and p_2 are even or they are both odd. If they are both even, then we have the equivalent inequalities

$$x^{n+1} \le t^{p_1} s^{p_2} w^2, \quad w^2 \le xu, \quad \text{or} \quad x^{\frac{n+1}{2}} \le t^{\frac{p_1}{2}} s^{\frac{p_2}{2}} w, \quad w^2 \le xu,$$

which is again of the form (5b). If both p_1 and p_2 are odd, then we introduce a few more variables, noting that inequality (5b) is equivalent to

$$x^{n+1} \le t^{p_1-1} s^{p_2-1} stux, \quad \text{or} \quad x^{n+1} \le t^{p_1-1} s^{p_2-1} y^4, \quad y^2 \le wv, \quad w^2 \le st, \quad v^2 \le ux,$$

and raising the inequality involving x^{n+1} to the power $\frac{1}{2}$ yields the four inequalities

$$x^{\frac{n+1}{2}} \le t^{\frac{p_1-1}{2}} s^{\frac{p_2-1}{2}} y^2, \quad y^2 \le wv, \quad w^2 \le st, \quad v^2 \le ux,$$

which is of the form (5c).

(3) The case u^2, inequality (5c). We show that we can always reduce this to one of the cases (5).

a. n is even: In this case, either both p_1 and p_2 are both even or they are both odd. If p_1 and p_2 are even, then inequality (5c) is equivalent to $x^{n/2} \le t^{p_1/2} s^{p_2/2} u$, which is of type (5b). If p_1 and p_2 are odd, then we have the equivalent representation

$$x^n \le t^{p_1-1} s^{p_2-1} u^2 st, \quad \text{or} \quad x^n \le t^{p_1-1} s^{p_2-1} y^4, \;\; y^4 \le u^2 w^2, \;\; w^2 \le st,$$

which (by inspection) is equivalent to the inequality of type (5c) (along with an additional two SOCP inequalities)

$$x^{\frac{n}{2}} \le t^{\frac{p_1-1}{2}} s^{\frac{p_2-1}{2}} y^2, \;\; y^2 \le uw, \;\; w^2 \le st.$$

b. n is odd: In this case, exactly one of p_1 and p_2 is odd; assume w.l.o.g. that p_1 is odd. Then inequality (5c) is equivalent to

$$x^{n+1} \le t^{p_1-1} s^{p_2} u^2 tx, \quad \text{or} \quad x^{\frac{n+1}{2}} \le t^{\frac{p_1-1}{2}} s^{\frac{p_2}{2}} y^2, \;\; y^2 \le uw, \;\; w^2 \le tx.$$

The preceding enumerated steps suggest a recursive strategy, where at each step, we check which of the cases we are in, then perform a reduction to introduce at most three inequalities of the form $w^2 \le uv$, and reducing the powers on all other variables by a factor of 2. The recursion halts as soon as we have an inequality of the form $x^n \le y^n$, or an inequality of the form $x^2 \le uv$, which is second-order-cone representable. Note that given an inequality of the form (4), we introduce at most $O(1) \lceil \log_2(p_1 \vee p_2) \rceil$ new inequalities via this recursion.

General (rational) p-norm cones as second-order cones

Now we show how to represent the inequality

$$\|x\|_p \le t, \quad \text{where } p = \frac{n}{m} \text{ for some } n, m \in \mathbb{N}, n \ge m \tag{6}$$

for $x \in \mathbb{R}^d$ as a collection of second-order cones and linear inequalities. First, note that the inequality

$$\left(\sum_{i=1}^d |x_i|^p \right)^{1/p} \le t \;\; \equiv \;\; \left(\sum_{i=1}^d |x_i|^{n/m} \right)^{m/n} \le t \;\; \equiv \;\; \sum_{i=1}^d t^{1-n/m} |x_i|^{n/m} \le t$$

is equivalent to the collection of inequalities $s^\top 1 \le t$, $|x_i|^{n/m} \le s_i t^{n/m-1}$ for all i, which in turn is equivalent to the set of inequalities

$$v_i \ge |x_i|, \; t \ge 0, \; s_i \ge 0, \; v_i^n \le s_i^m t^{n-m}, \; \sum_{i=1}^d s_i \le t.$$

We know this is SOCP representable by the recursions (5).

References

[1] F. Alizadeh and D. Goldfarb. Second-order cone programming. *Mathematical Programming, Series B*, 95:3–51, 2001.